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ON A DETERMINATION OF THE INITIAL FUNCTIONS FROM THE OBSERVED VALUES OF THE BOUNDARY FUNCTIONS FOR THE SECOND-ORDER HYPERBOLIC EQUATION

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Abstract. In this paper, the problem of determining the initial functions for the second order linear hyperbolic equation is considered. The problem is reduced to the optimal control problem. The necessary and sufficient optimality condition is derived for the obtained new problem.

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1 Introduction

In direct problems of mathematical physics, it is required to determine the functions describing various physical phenomena. In these cases, the coefficients of the equations of the studied processes, right-hand side, initial state of the process, boundary conditions are assumed to be known. However, these parameters are often unknown. Then inverse problems arise, in which, according to information about the solutions of the direct problem, it is required to determine certain characteristics of the process. These problems are usually ill-posed. Since these parameters are important characteristics of the process under consideration, the study of inverse problems for the equations of mathematical physics is one of the important problems of the modern applied mathematics (Kabankhin, 2009).

In this paper, the problem of determining the initial functions from the observable values of the boundary functions for the linear second-order hyperbolic equation is considered. This problem is reduced to the optimal control problem and is investigated by the methods of optimal control theory. We note that the similar problem for a parabolic equation was studied by Lions (1972).

2 Formulation of the problem

Let the state of the system $u(x,t)$ be described by the following second order hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + Au = f(x,t), \quad (x,t) \in Q, \tag{1}$$

where

$$Au \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j}) + a_0(x,t)u,$$

$Q = \Omega\times(0,T)$ is a cylinder in $R^{n+1}, \Omega \subset R^n$ is a bounded domain with smooth enough boundary; $T > 0$ is a given number, $f(x,t) \in L_2(Q)$, $a_{ij}(x,t) \in C^1(\bar{Q})$, $a_0(x,t) \in C(\bar{Q})$ are given functions and $a_{ij}(x,t) = a_{ji}(x,t), i,j = \overline{1,n}$,

$$\nu \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j; \nu = const > 0, \forall(x,t) \in \bar{Q}, \forall\xi = (\xi_1, \ldots, \xi_n) \in R^n.$$

Consider the problem of determining the initial state $(u(x,0), \frac{\partial u(x,0)}{\partial t})$ at the observed values of the state of the system at the boundary

$$u|_S = g_0, \frac{\partial u}{\partial \nu_A}|_S = g_1, \tag{2}$$

where $S = \Gamma \times (0,T)$ is a lateral surface of the cylinder Q, $\frac{\partial u}{\partial \nu_A} \equiv \sum\limits_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial u}{\partial x_j} \cos(\nu, x_i)$ is a conormal derivative; ν is outward normal to the boundary Γ of the domain $\Omega, g_0 \in W_2^1(S), g_1 \in L_2(S)$ are given functions.

It is supposed that there exists the function $u(x,t)$ from $W_2^1(Q)$, satisfying relations (1),(2). If $T > 0$ is large enough then such function is unique. Indeed if $\tilde{u}(x,t)$ is the second such function, then the difference $\bar{u} = u - \tilde{u}$ satisfies the relation

$$\frac{\partial^2 \bar{u}}{\partial t^2} + A\bar{u} = 0 \ \ in \ \ Q, \bar{u}|_S = 0, \frac{\partial u}{\partial \nu_A}|_S = 0,$$

for enough T $\bar{u} \equiv 0$ (T depends on the geometry of the domain Ω, see. Lattes et.al. (1970), p.196). Thus, there exists some defined initial state $(u(x,0), \frac{\partial u(x,0)}{\partial t})$. Consequently, one can set the problem on determination of the value $(u(x,0), \frac{\partial u(x,0)}{\partial t})$ of the solution to Cauchy problem (1),(2).

3 Transformation of the problem

We consider $u(x,0) = v_0^1(x), \frac{\partial u(x,0)}{\partial t} = v_0^2(x), v_0(x) = (v_0^1(x), v_0^2(x))$ as a control that should be defined by the optimal way in the sense given below. Let $v(x) = (v^1(x), v^2(x))$ be an arbitrary control, $v^1(x) \in \overset{\circ}{W}{}_2^1(\Omega), v^2(x) \in L_2(\Omega)$. Denote $H \equiv \overset{\circ}{W}{}_2^1(\Omega) \times L_2(\Omega)$.

Introduce two systems the states of which $u^1 = u^1(x,t;v)$ and $u^2 = u^2(x,t;v)$ are defined as solutions of the corresponding boundary value problems

$$\frac{\partial^2 u^1}{\partial t^2} + Au^1 = f \ \text{ in } \ Q, \tag{3}$$

$$u|_S = g_0, u^1(x,0;v) = v^1(x), \frac{\partial u^1(x,0;v)}{\partial t} = v^2(x) \ \text{ in } \ \Omega \tag{4}$$

and

$$\frac{\partial^2 u^2}{\partial t^2} + Au^2 = f \ \text{ in } \ Q, \tag{5}$$

$$\frac{\partial u}{\partial \nu_A}|_S = g_1, u^2(x,0;v) = v^1(x), \frac{\partial u^2(x,0;v)}{\partial t} = v^2(x) \ \text{ in } \ \Omega. \tag{6}$$

Here the compatibility condition

$$g_0|_{t=0} = v^1|_\Gamma = 0$$

should be met. Note that boundary value problem (3),(4) (also (5),(6)) for each control $v(x)$ from H has the only solution from $W_2^1(Q)$ (Ladyzhenskaya, 1973, pp.209-215; Lions et al., 1971, pp.296-302) and the following estimations are valid

$$\|u^1\|_{W_2^1(Q)} \leq c(\|f\|_{L_2(Q)} + \|v^1\|_{W_2^1(\Omega)} + \|v^2\|_{L_2(\Omega)} + \|g_0\|_{W_2^1(S)}) \tag{7}$$

and

$$\|u^2\|_{W_2^1(Q)} \leq c(\|f\|_{L_2(Q)} + \|v^1\|_{W_2^1(\Omega)} + \|v^2\|_{L_2(\Omega)} + \|g_1\|_{L_2(S)}). \tag{8}$$

Here and further on by c we denote different constants not depending on the admissible controls and quantities under estimation.

As a generalized solution of boundary value problem (3),(4) at given $v \in H$ we assume the function $u^2 \in W_2^1(Q)$, which is equal to $v^1(x)$ at $t = 0$ and satisfies to the integral equality

$$\int\limits_Q \left[-\frac{\partial u^1}{\partial t}\frac{\partial \varphi_1}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial u^1}{\partial x_j}\frac{\partial \varphi_1}{\partial x_i} + a_0 u\varphi \right] dxdt -$$

$$- \int\limits_\Omega v^2(x)\varphi_1(x,0)dx = \int\limits_Q f\varphi_1 dxdt,$$

for all $\varphi_1 = \varphi_1(x,t) \in W_{2,0}^1(Q), \varphi_1(x,t) = 0$. Similarly, the definition of the generalized solution to problem (5),(6) may be given. It is obvious that if $v(x) = v_0(x) = (v_0^1(x), v_0^2(x))$, then $u^1(v_0) = u^2(v_0)$. Introduce the functional

$$J(v) = \frac{1}{2}\int\limits_Q [u^1(v) - u^2(v)]^2 dxdt. \tag{9}$$

Then one can state that there exists a control $v_0 \in H$ for which

$$\inf_{v\in H} J(v) = J(v_0) = 0. \tag{10}$$

We search the value $(u(x,0) = v_0^1(x), \frac{\partial u(x,0)}{\partial t} = v_0^2(x))$ considering relation (10). Since relation (10) is equivalent to the initial problem, it contains the instability property of this problem. We reduce it to the stable problem. The functional $J(v)$ given by formula (9) is unconducive in the space H, therefore we have to regularize this functional. For $\varepsilon > 0$ consider the functional

$$J_\varepsilon(v) = J(v) + \frac{\varepsilon}{2}[\|v^1\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} + \|v^2\|^2_{L_2(\Omega)}], \tag{11}$$

that will be minimized in H. Since problems (3),(4) and (5),(6) are linear with respect to $v = (v^1, v^2)$, the functional $J(v)$ is convex, due to the second term $J_\varepsilon(v)$ is strong convex on H. Therefore, by virtue of the well-known theorem (Lions, 1972, p.13) in new problem (3)-(6),(11) there is a unique element $v_\varepsilon = (v_\varepsilon^1, v_\varepsilon^2) \in H$ minimizing $J_\varepsilon(v)$ and by virtue of the theorem (Lions, 1972, p.48) $v_\varepsilon \to v_0$ in H at $\varepsilon \to 0$. Thus, we have reduced the problem under consideration to the finding the element v_ε. This new problem is already stable. Indeed, relation (10) is valid only for some particular values g_0 and g_1, and satisfying the compatibility conditions. Therefore, the element $v_0(x)$ in relation (10) does not depend continuously on the functions g_0 and g_1. At the same time, the element v_ε depends continuously on these functions.

4 Search of v_ε and optimality condition

Consider more general problem: it needs to find $\inf J_\varepsilon(v)$ at $v \in U \subseteq H$, where U is a convex closed subset of H.

We investigate the Frechet differentiability of functional (11). Introduce the adjoint state $\psi = (\psi^1, \psi^2)$ setting $\psi^1 = \psi^1(x, t; v_\varepsilon)$, $\psi^2 = \psi^2(x, t; v_\varepsilon)$, as solutions of the problems

$$\frac{\partial^2 \psi^1}{\partial t^2} + A\psi^1 = u^1(v_\varepsilon) - u^2(v_\varepsilon) \text{ in } Q, \tag{12}$$

$$\psi^1|_S = 0, \psi^1(x, T; v_\varepsilon) = 0, \frac{\partial \psi^1(x, T; v_\varepsilon)}{\partial t} = 0 \text{ in } \Omega \tag{13}$$

and

$$\frac{\partial^2 \psi^2}{\partial t^2} + A\psi^2 = u^1(v_\varepsilon) - u^2(v_\varepsilon) \text{ in } Q, \tag{14}$$

$$\frac{\partial \psi^2}{\partial \nu_A}|_S = 0, \psi^2(x, T; v_\varepsilon) = 0, \frac{\partial \psi^2(x, T; v_\varepsilon)}{\partial t} = 0 \text{ in } \Omega. \tag{15}$$

As a generalized solutions from $W_2^1(Q)$ of boundary value problems (12),(13) and (14),(15) for the given $v_\varepsilon = (v_\varepsilon^1, v_\varepsilon^2) \in U$ we assume the vector function $\psi = (\psi^1, \psi^2) = (\psi^1(x, t; v_\varepsilon), \psi^2 = \psi^2(x, t; v_\varepsilon))$ equal to zero at $t = T$, components of which satisfy the integral equalities below

$$\int_Q \left[-\frac{\partial \psi^1}{\partial t}\frac{\partial \eta_1}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial \psi^1}{\partial x_j}\frac{\partial \eta_1}{\partial x_i} + a_0\psi^1\eta_1 \right] dxdt-$$

$$-\int_\Omega \frac{\partial \psi^1(x, 0; v_\varepsilon)}{\partial t}\eta_1(x, 0)dx = \int_Q [u^1(v_\varepsilon) - u^2(v_\varepsilon)]\eta_1 dxdt, \tag{16}$$

for all $\eta_1 = \eta_1 \in W_{2,0}^1(Q)$,

$$\int_Q \left[-\frac{\partial \psi^2}{\partial t}\frac{\partial \eta_2}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial \psi^2}{\partial x_j}\frac{\partial \eta_2}{\partial x_i} + a_0\psi^2\eta_2 \right] dxdt-$$

$$-\int_\Omega \frac{\partial \psi^2(x, 0; v_\varepsilon)}{\partial t}\eta_2(x, 0)dx = \int_Q [u^1(v_\varepsilon) - u^2(v_\varepsilon)]\eta_2 dxdt \tag{17}$$

for all $\eta_2 = \eta_2 \in W_2^1(Q)$.

As follows from Ladyzhenskaya (1973), p.209-215; Lions et.al. (1971), p.296-302 under the above assumptions for each given $v_\varepsilon \in U$ each of problems (12),(13) and (14),(15) has the only solution from $W_2^1(Q)$ and the estimations

$$\|\psi^1\|_{W_2^1(Q)} \le c\|u^1(v_\varepsilon) - u^2(v_\varepsilon)\|_{L_2(Q)},$$

$$\|\psi^2\|_{W_2^1(Q)} \le c\|u^1(v_\varepsilon) - u^2(v_\varepsilon)\|_{L_2(Q)}$$

are valid. Then from those estimation and also from (7),(8) follows that

$$\|\psi^1\|_{W_2^1(Q)} \le c(\|f\|_{L_2(Q)} + \|v_\varepsilon^1\|_{W_2^1(\Omega)} + \|v_\varepsilon^2\|_{L_2(\Omega)} + \|g_0\|_{W_2^1(S)} + \|g_1\|_{L_2(S)}) \tag{18}$$

and

$$\|\psi^2\|_{W_2^1(Q)} \le c(\|f\|_{L_2(Q)} + \|v_\varepsilon^1\|_{W_2^1(\Omega)} + \|v_\varepsilon^2\|_{L_2(\Omega)} + \|g_0\|_{W_2^1(S)} + \|g_1\|_{L_2(S)}). \tag{19}$$

Theorem 1. *Let the conditions of problems* (3)-(6),(11) *be satisfied. Then functional* (11) *is continuously Frechet differentiable on* U *and its differential in the point* $v_\varepsilon \in U$ *at the increment* $\delta v = (\delta v^1, \delta v^2) \in H, v_\varepsilon + \delta v \in U$ *is defined by the expression*

$$< J_\varepsilon'(v_\varepsilon), \delta v >= \int_\Omega [\psi^1(x, 0; v_\varepsilon) - \psi^2(x, 0; v_\varepsilon)]\delta v^2(x)dx+$$

$$+\int_{\Omega}[\frac{\partial\psi^1(x,0;v_\varepsilon)}{\partial t}-\frac{\psi^2(x,0;v_\varepsilon)}{\partial t}]\delta v^1(x)dx+$$

$$+\varepsilon\int_{\Omega}[v_\varepsilon^1\delta v^1+\sum_{i=1}^n\frac{\partial v_\varepsilon^1}{\partial x_i}\frac{\partial\delta v^1}{\partial x_i}+v_\varepsilon^2\delta v^2]dx. \tag{20}$$

Proof. Let $v_\varepsilon, v_\varepsilon+\delta v\in U$ be arbitrary controls and $\delta u=(\delta u^1,\delta u^2)$, $\delta u^1=\delta u^1(x,t)=u^1(x,t;v_\varepsilon+\delta v)-u^1(x,t;v_\varepsilon)$, $\delta u^2=\delta u^2(x,t)=u^2(x,t;v_\varepsilon+\delta v)-u^2(x,t;v_\varepsilon)$. From (3)-(6) follows that $\delta u^1,\delta u^2$ are solutions from $W_2^1(Q)$ to the following boundary value problems

$$\frac{\partial^2\delta u^1}{\partial t^2}+A\delta u^1=0\ \text{ in }\ Q, \tag{21}$$

$$\delta u^1|_S=,\delta u^1(x,0)=\delta v^1(x),\frac{\partial\delta u^1(x,0)}{\partial t}=\delta v^2(x)\ \text{ in }\ \Omega \tag{22}$$

and

$$\frac{\partial^2\delta u^2}{\partial t^2}+A\delta u^2=0\ \text{ in }\ Q, \tag{23}$$

$$\frac{\partial\delta u}{\partial\nu_A}|_S=0,\delta u^2(x,0)=\delta v^1(x),\frac{\partial\delta u^2(x,0)}{\partial t}=\delta v^2(x)\ \text{ in }\ \Omega. \tag{24}$$

From the knows results (Ladyzheskaya, 1973, pp.209-215; Lions et al., 1971, pp.296-302) we obtain that for the solution of problems (21),(22) and (23),(24) the estimations

$$\|\delta u^1\|_{W_2^1(Q)}\le c(\|\delta v^1\|_{\overset{\circ}{W}{}_2^1(\Omega)}+\|\delta v^2\|_{L_2(\Omega)}), \tag{25}$$

$$\|\delta u^2\|_{W_2^1(Q)}\le c(\|\delta v^1\|_{\overset{\circ}{W}{}_2^1(\Omega)}+\|\delta v^2\|_{L_2(\Omega)}). \tag{26}$$

are valid.

Increment $\Delta J_\varepsilon(v_\varepsilon)=J_\varepsilon(v_\varepsilon+\delta v)-J_\varepsilon(v_\varepsilon)$ of functional (11) in the point $v_\varepsilon\in U$ has a form

$$\Delta J_\varepsilon(v_\varepsilon)=\frac{1}{2}\int_Q\left\{[u^1(v_\varepsilon+\delta v)-u^2(v_\varepsilon+\delta v)]^2-[u^1(v_\varepsilon)-u^2(v_\varepsilon)]^2\right\}dxdt+$$

$$+\frac{\varepsilon}{2}\int_{\Omega}\left[(v_\varepsilon^1+\delta v^1)^2-(v_\varepsilon^1)^2+\sum_{i=1}^n\left[\left(\frac{\partial(v_\varepsilon^1+\delta v^1)}{\partial x_i}\right)^2-\left(\frac{\partial v_\varepsilon^1}{\partial x_i}\right)^2\right]+\right.$$

$$\left.+\ (v_\varepsilon^2+\delta v^2)^2-(v_\varepsilon^2)^2\right]dx=\int_Q[u^1(v_\varepsilon)-u^2(v_\varepsilon)](\delta u^1-\delta u^2)dxdt+$$

$$+\varepsilon\int_{\Omega}[v_\varepsilon^1\delta v^1+\sum_{i=1}^n\frac{\partial v_\varepsilon^1}{\partial x_i}\frac{\partial\delta v^1}{\partial x_i}+v_\varepsilon^2\delta v^2]+$$

$$+\frac{1}{2}\int_Q(\delta u^1-\delta u^2)^2dxdt+\frac{\varepsilon}{2}\int_{\Omega}[(\delta v^1)^2+\sum_{i=1}^n(\frac{\partial\delta v^1}{\partial x_i})^2+(\delta v^2)^2]dx. \tag{27}$$

It is obvious that the functions $\delta u^1(x,t)$ and $\delta u^2(x,t)$ satisfy following integral identities

$$\int_Q\left[-\frac{\partial\delta u^1}{\partial t}\frac{\partial\varphi_1}{\partial t}+\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial\delta u^1}{\partial x_j}\frac{\partial\varphi_1}{\partial x_i}+a_0\delta u^1\varphi_1\right]dxdt-$$

$$-\int_{\Omega}\varphi_1(x,0)\delta v^2 dx = 0 \tag{28}$$

for all $\varphi_1 \in W_{2,0}^1(Q)$, $\varphi_1(x,T)=0$,

$$\int_Q \left[-\frac{\partial\delta u^2}{\partial t}\frac{\partial\varphi_2}{\partial t} + \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial\delta u^2}{\partial x_j}\frac{\partial\varphi_2}{\partial x_i} + a_0\delta u^2\varphi_2\right] dxdt-$$

$$-\int_{\Omega}\varphi_2(x,0)\delta v^2 dx = 0, \tag{29}$$

for all $\varphi_2 \in W_2^1(Q)$, $\varphi_2(x,T)=0$ and conditions $\delta u^1(x,0)=\delta v^1(x)$, $\delta u^2(x,0)=\delta v^1(x)$ at $t=0$. Taking $\eta_1=\delta u^1$, $\eta_2=\delta u^2$ in identities (16), (17) and $\varphi_1=\psi^1$, $\varphi_2=\psi^2$ in (28), (29) and then subtracting (28) from (16) and (29) from (17) we get

$$-\int_{\Omega}\frac{\partial\psi^1(x,0)}{\partial t}\delta v^1(x)dx + \int_{\Omega}\psi^1(x,0)\delta v^2(x)dx = \int_Q [u^1(v_\varepsilon)-u^2(v_\varepsilon)]\delta u^1 dxdt, \tag{30}$$

$$-\int_{\Omega}\frac{\partial\psi^2(x,0)}{\partial t}\delta v^1(x)dx + \int_{\Omega}\psi^2(x,0)\delta v^2(x)dx = \int_Q [u^1(v_\varepsilon)-u^2(v_\varepsilon)]\delta u^2 dxdt. \tag{31}$$

From formulas (30) and (31) follows that

$$\int_Q [u^1(v_\varepsilon)-u^2(v_\varepsilon)][\delta u^1-\delta u^2]dxdt =$$

$$=\int_{\Omega}\left[\frac{\partial\psi^2(x,0)}{\partial t}-\frac{\partial\psi^1(x,0)}{\partial t}\right]\delta v^1(x)dx + \int_{\Omega}\left[\psi^1(x,0)-\psi^2(x,0)\right]\delta v^2(x)dx. \tag{32}$$

If to consider formula (32) in the expression for the increment of functional (27), then we obtain

$$\Delta J_\varepsilon(v_\varepsilon) = \int_{\Omega}\left[\frac{\partial\psi^2(x,0)}{\partial t}-\frac{\partial\psi^1(x,0)}{\partial t}\right]\delta v^1(x)dx+$$

$$+\int_{\Omega}\left[\psi^1(x,0)-\psi^2(x,0)\right]\delta v^2(x)dx + \varepsilon\int_{\Omega}\left[v_\varepsilon^1\delta v^1+\right.$$

$$\left.+\sum_{i=1}^n \frac{\partial v_\varepsilon^1}{\partial x_i}\frac{\partial\delta v^1}{\partial x_i} + v_\varepsilon^2\delta v^2\right]dx + R, \tag{33}$$

where

$$R = \frac{1}{2}\int_Q(\delta u^1-\delta u^2)^2 dxdt + \frac{\varepsilon}{2}\int_{\Omega}[(\delta v^1)^2 + \sum_{i=1}^n(\frac{\partial\delta v^1}{\partial x_i})^2 + (\delta v^2)^2]dx \tag{34}$$

is a remainder term. From estimations (25), (26) and expression for R it is easy to get the estimation

$$R \le c(\|\delta v^1\|_{\overset{\circ}{W}{}_2^1(\Omega)} + \|\delta v^2\|_{L_2(\Omega)}). \tag{35}$$

Thus from formula (33) and estimation (35) follows that the functional $J_\varepsilon(v)$ is Frechet differentiable on U and its differential is determined by expression (20).

Now we show that the mapping $v_\varepsilon \to J'_\varepsilon(v_\varepsilon)$, defined by expression (20) acts continuously from U into H^*, where H^* is adjoint to the space H and $H^* \equiv W_2^{-1}(\Omega)\times L_2(\Omega)$. Let $\delta\psi =$

$(\delta\psi^1, \delta\psi^2) = (\psi^1(x,t;v_\varepsilon+\delta v_\varepsilon) - \psi^1(x,t;v_\varepsilon), \psi^2(x,t;v_\varepsilon+\delta v_\varepsilon) - \psi^2(x,t;v_\varepsilon))$. From (12)-(15) follows that $\delta\psi^1$ and $\delta\psi^2$ are solutions from the class $W_2^1(Q)$ to the boundary value problems

$$\frac{\partial^2 \delta\psi^1}{\partial t^2} + A\delta\psi^1 = \delta u^1(v_\varepsilon) - \delta u^2(v_\varepsilon) \text{ in } Q,$$

$$\delta\psi|_S = 0, \delta\psi^1(x,T;v_\varepsilon) = 0, \frac{\partial \delta\psi^1(x,T;v_\varepsilon)}{\partial t} = 0 \text{ in } \Omega$$

and

$$\frac{\partial^2 \delta\psi^2}{\partial t^2} + A\delta\psi^2 = \delta u^1(v_\varepsilon) - \delta u^2(v_\varepsilon) \text{ in } Q,$$

$$\frac{\partial \delta\psi}{\partial \nu_A}|_S = 0, \delta\psi^2(x,T;v_\varepsilon) = 0, \frac{\partial \delta\psi^2(x,T;v_\varepsilon)}{\partial t} = 0 \text{ in } \Omega.$$

As in (18), (19), for the solutions to these problems the estimations

$$\|\delta\psi^1\|_{W_2^1(Q)} \le c(\|\delta u^1\|_{L_2(Q)} + \|\delta u^2\|_{L_2(Q)}) \le c(\|\delta v_\varepsilon^1\|_{W_2^1(\Omega)} + \|\delta v_\varepsilon^2\|_{L_2(\Omega)}),$$

$$\|\delta\psi^2\|_{W_2^1(Q)} \le c(\|\delta u^1\|_{L_2(Q)} + \|\delta u^2\|_{L_2(Q)}) \le c(\|\delta v_\varepsilon^1\|_{W_2^1(\Omega)} + \|\delta v_\varepsilon^2\|_{L_2(\Omega)}). \tag{36}$$

are valid. Moreover (20) implies validity of the inequality

$$\|J'_\varepsilon(v_\varepsilon+\delta v_\varepsilon) - J'_\varepsilon(v)\|_{H^*} \le c\int_\Omega \Big[|\delta\psi^1(x,0;v_\varepsilon)| + |\delta\psi^2(x,0;v_\varepsilon)| +$$

$$+|\frac{\partial \delta\psi^1(x,0;v_\varepsilon)}{\partial t}| + |\frac{\partial \delta\psi^2(x,0;v_\varepsilon)}{\partial t}|\Big] dx +$$

$$+c\int_\Omega [|\delta v_\varepsilon^1| + \sum_{i=1}^n |\frac{\partial \delta v_\varepsilon^1}{\partial x_i}| + |\delta v_\varepsilon^2|]dx.$$

Then due to estimation (36) the right hand side of this inequality tends to zero at $\|\delta v_\varepsilon\|_H \to 0$. It leads us to the fact that $v_\varepsilon \to J'_\varepsilon(v_\varepsilon)$ is a continuous mapping from U into H^*. Theorem 1 is proved. □

The next theorem is on the necessary and sufficient optimality conditions in problem (3)-(6),(11) using the differential of functional (11).

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then for the optimality of the control*$v_\varepsilon(x) = (v_\varepsilon^1(x), v_\varepsilon^2(x)) \in U$ *in problem* (3)-(6),(11) *it is necessary and sufficient fulfilment of the inequality*

$$\int_\Omega [\psi^1(x,0;v_\varepsilon) - \psi^2(x,0;v_\varepsilon)](v^2(x) - v_\varepsilon^2(x))dx +$$

$$+\int_\Omega [\frac{\partial \psi^1(x,0;v_\varepsilon)}{\partial t} - \frac{\psi^2(x,0;v_\varepsilon)}{\partial t}](v^1(x) - v_\varepsilon^1(x))dx +$$

$$+\varepsilon\int_\Omega [v_\varepsilon^1(v^1(x) - v_\varepsilon^1(x)) + \sum_{i=1}^n \frac{\partial v_\varepsilon^1}{\partial x_i}\left(\frac{\partial v^1(x)}{\partial x_i} - \frac{\partial v_\varepsilon^1(x)}{\partial x_i}\right) + v_\varepsilon^2(v^2(x) - v_\varepsilon^2(x))] \ge 0, \tag{37}$$

for arbitrary $v = v(x) = (v^1(x), v^2(x)) \in U$.

Proof. The set U is convex in H and the functional $J_\varepsilon(v)$ is strongly convex on U and according to Theorem 1 functional (11) is continuously Frechet differentiable on U. Then due to Theorem 5 (Vasilyev, 1981, p.28)on the element $v_\varepsilon \in U$ it is necessary and sufficient fulfilment of the inequality $< J'_\varepsilon(v_\varepsilon), v - v_\varepsilon >$ for all $v \in U$. From this and (20) follows the validity of inequality (37). Theorem 2 is proved. □

Now if to suppose that $U = H$ in condition (37), then we get

$$\psi^1(x,0;v_\varepsilon) - \psi^2(x,0;v_\varepsilon) + \varepsilon v_\varepsilon^2(x) = 0 \text{ in } \Omega,$$

$$\frac{\partial\psi^1(x,0;v_\varepsilon)}{\partial t} - \frac{\psi^2(x,0;v_\varepsilon)}{\partial t} + \varepsilon(v_\varepsilon^1(x) - \Delta v_\varepsilon^1(x)) = 0 \text{ in } \Omega$$

in the sense of distributions, where Δ is the Laplace operator.

As a result we get the following

Theorem 3. *Let the conditions in the formulation of problem* (3)-(6),(11) *hold. Then to find the optimal control* $v_\varepsilon(x) = (v_\varepsilon^1(x), v_\varepsilon^2(x)) \in H$ *it is necessary solving the problem:*

$$\begin{cases} \frac{\partial^2 u^1}{\partial t^2} + Au^1 = f, \frac{\partial^2 u^2}{\partial t^2} + Au^2 = f, \\ \frac{\partial^2 \psi^1}{\partial t^2} + A\psi^1 = u^1 - u^2, \frac{\partial^2 \psi^2}{\partial t^2} + A\psi^2 = u^1 - u^2 \ \textit{in } Q; \end{cases} \tag{38}$$

$$u^1 = g_0, \frac{\partial u^2}{\partial \nu_A} = g_1, \psi^1 = 0, \frac{\partial \psi^2}{\partial \nu_A} = 0 \ \textit{on } S; \tag{39}$$

$$\begin{cases} u^1(x,0) = u^2(x,0), \frac{\partial u^1(x,0)}{\partial t} = \frac{\partial u^2(x,0)}{\partial t}, \\ \psi^1(x,T) = 0, \frac{\partial \psi^1(x,T)}{\partial t} = 0, \psi^2(x,T) = 0, \frac{\partial \psi^2(x,T)}{\partial t} = 0, \\ \psi^1(x,0) - \psi^2(x,0) + \varepsilon\frac{\partial u^1(x,0)}{\partial t} = 0 \ \textit{in } \Omega, \frac{\partial \psi^1(x,0)}{\partial t} - \frac{\psi^2(x,0)}{\partial t} + \varepsilon(u^1(x,0) - \Delta u^1(x,0)) = 0 \ \textit{in } \Omega, \end{cases} \tag{40}$$

in the sense of distributions.

It follows from this fact that, to find the optimal control $v_\varepsilon(x)$ one first have to solve system (38)-(40), and then set

$$v_\varepsilon^1(x) = u_\varepsilon^1(x,0), v_\varepsilon^2(x) = \frac{\partial u_\varepsilon^1(x,0)}{\partial t}.$$

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